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This material has been prepared to provide access to information and to encourage discussion that can inform research, policy, and practice. The information contained in this document should not be used in isolation to reach definitive conclusions. REL Northwest staff are available to facilitate discussion, to provide further relevant information, and, in some cases, to partner on research to build an increasingly solid body of knowledge.

Handout B: Literature Summary—Process Standards

Review Summary

The Montana Office of Public Instruction (OPI) developed state level content standards for mathematics in 2011, which they now seek to revise and update. OPI asked REL Northwest to summarize recent research about how to build a rigorous mathematics program incorporating foundational mathematical ideas for students in Grades K–12. Findings from the review are as follows.

Theme 1: How do current standards address the importance of developing and integrating mathematics process standards alongside content standards? (p. 2 – 5)

- Routines, norms, and habits of mind are central to understanding and using mathematics.
- Although the Standards for Mathematical Practice are currently in broad use across the U.S., recently developed National Assessment of Educational Progress (NAEP) standards reflect an evolution of process standards.

Theme 2. What does the research evidence say about developing conceptual and procedural knowledge in mathematics in relation to process standards? (p. 6 – 11)

Building and regularly assessing students' mathematical processes or practices support students' conceptual and procedural knowledge. Specific recommendations include

- teach students how to use and connect visual representations;
- engage students with multiple problem-solving strategies;
- support mathematical discussions;
- help students to monitor and reflect on problem-solving using feedback, worked examples, and error reflection; and
- scaffold learning with explicit instructional support that gradually fades to encourage independent problem-solving.

Theme 3. What does the research evidence say about the impact of process standards on equity, particularly in the contexts of American Indian communities? (p. 12 – 13)

- support educators to directly address issues of race, power, and equity to support their students' learning needs
- use culturally relevant pedagogies
- consider students' social–emotional well-being

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Why This Review?

Recent research has highlighted the importance of process standards that require students to not only know how to solve a problem but also understand why those problem-solving steps are appropriate (Bisra et al., 2018). Process standards are intertwined with all content standards and typically include methods such as problem-solving strategies, using reasoning and proof, communicating math concepts, making connections between various types of mathematical problems, and creating various representations of math concepts (NCTM, 2000). OPI is particularly interested in learning about the research evidence supporting the efficacy of process standards, evidence about how students develop conceptual and procedural knowledge, and research on practices that support students' attainment of process standards.

Organization of the Review

To respond to its OPI request, this report presents three themes related to mathematics process standards, drawing primarily on research studies from 2010 and later.

Here are the themes, stated as research questions, that drove our literature review:

- Theme 1. How do current standards address the importance of developing and integrating mathematics process standards alongside content standards? (p. 2-5)
- Theme 2. What does the research evidence say about developing conceptual and procedural knowledge in mathematics in relation to process standards? (p. 6-11)
- Theme 3. What does the research evidence say about the impact of process standards on equity, particularly in the contexts of American Indian communities? (p. 12-13)

First, we provide an overview of mathematical practices and process standards, many of which are now integrated into or reflected by states' current math content standards (Friedberg et al., 2018; O'Keefe & Lewis, 2019). Next, we review the trends in research regarding the development of conceptual and procedural knowledge, since these two types of knowledge are central to mathematical proficiency and what the process standards are ultimately designed to promote. Finally, we summarize literature regarding the role of equity in the development and implementation of process standards. We pay particular attention to summarizing the available research related to standards-based approaches in Indigenous[1] communities.

[1] Native American, American Indian, Native, and Indigenous are terms used throughout this text to refer to Indigenous peoples of the United States. Different authors use different terms, and we use the terms from the original source. Each of these are imperfect, general terms.

Theme 1: How do current standards address the importance of developing and integrating mathematics process standards alongside content standards?

Routines, norms, and habits of mind are central to understanding and using mathematics.

- Mathematical processes and practices are essential to using math and are reflected in math standards endorsed by math educators (see Table 1).

Table 1. The Evolution of Mathematical Processes and Practices (2000–20)

Standards for Mathematical Processes (NCTM, 2000)	Strands of Mathematical Proficiency (NRC, 2001)	Standards for Mathematical Practice (CCSSI, 2010)	NAEP Mathematical Practices (NAGB, 2020)
• problem-solving • reasoning and proof • communication • connections • representations	• adaptive reasoning • strategic competence • conceptual understanding • procedural fluency • productive disposition	• Make sense of problems and persevere in solving them. • Reason abstractly and quantitatively. • Construct viable arguments and critique the reasoning of others. • Model with mathematics. • Use appropriate tools strategically. • Attend to precision. • Look for and make use of structure. • Look for and express regularity in repeated reasoning.	• representing • abstracting and generalizing • justifying and proving • mathematical modeling • collaborative mathematics

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The Common Core State Standards for Mathematics (CCSSM) published in 2010 included Standards for Mathematical Practice (SMPs).

- Forty-five states, including Montana (Montana Office of Public Instruction, 2011), have adopted the Common Core Standards for Mathematical Practice with Standards for Mathematical Practice (CCSSI, 2010). These standards include
  - Make sense of problems and persevere in solving them.
  - Reason abstractly and quantitatively.
  - Construct viable arguments and critique the reasoning of others.
  - Model with mathematics.
  - Use appropriate tools strategically.
  - Attend to precision.
  - Look for and make use of structure.
  - Look for and express regularity in repeated reasoning.
- Most states' math standards include Standards for Mathematical Practice (Friedberg et al., 2018; O'Keefe & Lewis, 2019).

Although the Standards for Mathematical Practice are currently in broad use across the U.S., recently developed NAEP standards reflect an evolution of process standards.

- For example, the recent report by the National Assessment Governing Board (2020), describes five mathematical practices that are driving the development of the 2025 National Assessment of Educational Progress (NAEP) mathematics assessment. The practices include
  - **Representing**. Recognizing, using, creating, interpreting, or translating among representations appropriate for the grade level and the mathematics being assessed.
  - **Abstracting and generalizing**. Decontextualizing, identifying commonality across cases, items, problems, or representations, and extending one's reasoning to a broader domain appropriate for the grade level and the mathematics being assessed.
  - **Justifying and proving**. Creating, evaluating, showing, proving, or refuting mathematical arguments and suppositions in developmentally and mathematically appropriate ways.
  - **Mathematical modeling**. Making sense of a scenario, identifying a problem to be solved, mathematizing it, and applying the mathematization to reach a solution and checking the viability of the solution in developmentally and mathematically appropriate ways.
  - **Collaborative mathematics**. The social enterprise of doing mathematics with others through discussion and collaborative problem-solving whereby ideas are offered, debated, connected, and built-upon toward solutions and shared understanding. Collaborative mathematics involves joint thinking among

individuals toward the construction of a problem–solution in developmentally and mathematically appropriate ways.

- NAEP includes assessment items at three levels of complexity: low, moderate, and high (National Assessment Governing Board, 2022). The higher the level of complexity, the more flexibility in thinking and problem-solving is expected.

Theme 2: What does the research evidence say about developing conceptual and procedural knowledge in mathematics in relation to process standards?

Background

The evidence base for instructional practices that promote conceptual understanding, procedural fluency, and problem-solving has grown significantly over the past decade. A considerable amount of this evidence is generated through individual studies and research syntheses funded by IES. This evidence base will further expand as results from studies recently conducted or currently in the field are published.

Conceptual and procedural knowledge are strongly related, and both are critical for mathematical proficiency.

- Math proficiency involves both conceptual knowledge (a deep understanding of math concepts and how they are connected) and procedural knowledge (how to carry out mathematical procedures) (Star et al., 2015; Woodward et al., 2012). These types of knowledge are developed through instructional practices that require students to both solve problems and understand the rationale behind problem-solving steps, such as having students analyze and learn from worked examples of problems as well as practicing their own thinking.
- Procedural knowledge supports conceptual knowledge, as well as vice versa (Hurrell, 2021; Hussein & Csikos, 2023; Kapur, 2014; Rittle-Johnson, et al., 2015; Rittle-Johnson & Schneider, 2015). For example, conceptual knowledge can help learners identify appropriate procedures and improve the ability to generalize procedures to novel problems (Crooks & Alibali, 2014).

Foundational Content

Teach students how to link between concrete and abstract representations

- Mathematics requires students to understand many different visual representations including pictures, symbolic notations, graphs, and pictures. Concrete representations (for example, a physical object or a depiction of a familiar item) help students meaningfully connect to prior knowledge and abstract representations can help students transfer knowledge to other situations (Belenky & Schalk, 2014).
- As students advance across the grade levels, students can benefit from continuing to use familiar (often concrete) representations and connecting those to increasingly more complex visual representations (Woodward et al., 2012). For example, students can begin to use the number line in elementary school to learn about whole numbers and fractions and later in middle school to learn about and compare the size of irrational numbers (Common Core Standards Writing Team, 2023).

Teach with multiple representations

- Visual representations are useful for both general education students and students with learning disabilities (Lambert & Sugita, 2016; Moschkovich, 1999).
- Since students develop mathematical understandings at different rates and have different exposure to representations, to help students develop understanding, teachers should
    - assess their students' knowledge of representations;
    - provide instruction on representations to help students select and use representations that are appropriate for the problems they are solving and to translate quantitative information into symbolic equations;
    - help students learn about multiple visual representations and be exposed to how other students use them, acknowledging that too many representations may be overwhelming and unhelpful for students (Woodward, et al., 2012); and
    - demonstrate how to represent a problem with visual representations—"thinking aloud" to describe decisions they are making—and how parts of the visual representation can be translated into mathematical notation.

Connect algebraic equations with visual representations

- Middle and high school students should learn that different algebraic representations can convey different information about an algebra problem:
    - "Recognizing and explaining corresponding features of the structure of two representations can help students understand the relationships among several algebraic representations, such as equations, graphs, and word problems" (Star et al., 2015, p 20).

Provide students with examples of different solutions to problems

- Expose students to different problem-solving strategies—either taught by the teacher or learned through peer sharing, comparing, or having students intentionally chose from alternate algebraic (problem-solving) strategies (Star et al., 2015; Woodward et al., 2012). This supports their flexible reasoning and ability to use different non-algorithmic strategies when they encounter an unfamiliar problem.
- Teachers can support student problem-solving efficiency by emphasizing the clarity and efficiency of different strategies for different mathematical situations. Teachers' use of unsuccessful strategies and alternate strategies can also help demonstrate that problems are not always solved easily the first time and that sometimes problem solvers need to try more than one strategy to solve a problem (that is, Standard for Mathematical Practice 1) (Woodward et al., 2012).

Consider and vary the cognitive demand of problem-solving activities

- Different mathematical activities require different levels of cognitive demand. For example, solving algebra problems often requires abstract thinking and the ability to process multiple pieces of complex information simultaneously. This can limit students' capacity to develop new knowledge. "Solved problems can minimize the burden of abstract reasoning by allowing students to see the problem and many solution steps at once—without executing each step—helping students learn more efficiently" (Star et al., 2015, p. 4).
- Teachers' explicit use of routine and non-routine problems (where the solution method is not predictable or already well-known to the student) can build student fluency and flexibility in thinking (Woodward et al., 2012).
    - Routine problems may help students develop deeper meaning of a mathematical concept, or extend their prior knowledge to more difficult, multistep problems.
    - Non-routine problems may help develop students' ability to think strategically and "force students to apply what they have learned in a new way" (Woodward et al., 2012, p. 12).
    - Non-routine problems often increase the cognitive demand for students and may require more instructional time for students to investigate and reason about the problem.

Support mathematical discussions

- To be successful problem solvers, students need to understand what is being asked of them. Problems discussed during whole class discussions can help students to leverage the diverse knowledge within the classroom (Woodward et al., 2012).
- Instructional routines such as Three-Read protocols are increasingly popular to enable students to support collaborative sense-making, although there is still limited empirical evidence of their effectiveness (McCoy, 2023; see Kelemanik et al., 2016, for a description of the routine). In this routine, students read a math problem three times with a different goal each time. The first read is to understand the context. The second read is to understand the mathematics or what is the mathematical question. The third read is to identify valuable information in the problem and elicit inquiry questions based on the scenario.
- Encourage students to compare and discuss multiple strategies. Students learn more when they compare multiple strategies, work in small groups, and engage in mathematical discussions (Durkin et al., 2023).

Provide students with timely feedback

- Specific, process-oriented, elaborative feedback that provides individualized information about students' strengths and weaknesses promote learning more than assigning a grade or merely indicating correctness (Mertens et al., 2022).
- Immediate feedback appears to be more effective than delayed feedback (Kehrer et al., 2013).

Have students study and explain both correct and incorrect worked examples

- Students may learn more when they alternate between studying examples of worked-out problem solutions and generating their own thinking or solving similar problems on their own than they do when just given problems to solve on their own (Booth et al., 2013; Pashler et al., 2007; Star et al., 2015). Worked examples may help students focus on the mathematical structure—the underlying mathematical features and relationships of an expression, representation, or equation—within the problems.
- Students with low prior knowledge benefit when asked to explain why incorrect solutions are incorrect (Barbieri & Booth, 2016).

Ask students to monitor and reflect on problem-solving

- Asking students to monitor and reflect on problem-solving may encourage them to answer important questions such as "What is the question asking me to do?" and "Why did these steps in solving the problem work or not work?" (Woodward et al., 2012).
- Meta-cognitive thinking and reflection can help students understand what kinds of strategies work for particular kinds of problems and make connections from their existing mathematical knowledge to new situations.
- With practice, students can use what they learn to classify types of problems and how they might be solved and then use that knowledge to master multistep or increasingly more complex problems.

Provide explicit instruction and scaffolding on the problem-solving process

- Because problem-solving steps may not always be clear and students solve problems in diverse ways, students up through middle school can benefit when teachers provide explicit instruction and some scaffolding on the problem-solving process (Woodward et al., 2012).
- Example scaffolds might include
  - task lists that identified specific steps to solving problems,
  - self-questioning checklists,
  - using visual aids or emphasizing multiple strategies, and
  - teacher modeling of a self-questioning process during the lesson.

Tailor instruction to the grade level and prior knowledge of students

- International studies have depicted U.S. classroom teachers frequently following a lesson format where teachers demonstrate or provide direct instruction related to a problem and then have students do the problem themselves (Stigler & Hiebert, 1999; Stigler & Hiebert, 2009).
  - Instruction followed by problem-solving may be effective when younger students (Grades 2–5) are learning a new concept (Sinha & Kapur, 2021).
  - Older students (Grades 6–10) may be better served by engaging in problem-solving activities first when they are learning a new concept.

Theme 3: What does the research evidence say about the impact of process standards on equity, particularly in the contexts of American Indian communities?

The scan of the literature revealed a lack of empirical research supporting specific approaches to improving outcomes for American Indian students. However, the available empirical evidence suggests that reforms are likely to be aligned with the CCSSM and process standards, which broadly seek to create equitable math classrooms that are engaging and relevant to all students (Barajas-Lopéz & Larnell, 2019; Bartell, 2017). Future studies might build on the following emerging ideas.

Support educators to attend directly to issues of race, power, and equity to support their students' learning needs.

- Teacher professional development may not have the content and focus (on race) needed to equitably impact student outcomes, especially when organized by White educators who may be culturally mis-aligned with their students. A race-neutral or "cultureless" approach to teacher professional development may limit its ability to equitably impact student results (Melhuis et al., 2022).
- Teachers may benefit from learning about and attending to the local, sociocultural, and historical aspects of mathematics classrooms. Teacher noticing[2] (van Es et al., 2022) determines which students are invited to participate, who is valued, and whose forms of knowing are included in mathematics classrooms. Each of these are relevant factors foundational to student opportunities to engage in mathematical practices and in creating equitable learning spaces for students.
- Power dynamics in the classroom may affect students, determine educational experiences, and create inequitable opportunities for students' knowledge development. Five levels of power that organize (in)equity in mathematics education include individuals, moment-to-moment interactions, local settings, interrelations between local settings, and extra local structures (Matthews et al., 2022).
- Especially for Indigenous student populations, there may be implications for mathematics education of Indigenous "futurity" (wherein even small actions taken now matter for a "hopeful future," p. 380) and spirituality (for example, relationships to lands and waterways and learning from more-than-human relatives) (Gutiérrez, 2022).

Use culturally relevant pedagogies.

Further studies might focus on making instruction more culturally relevant for American Indian students. These might include opportunities for teachers from similar cultural

[2] Sherin and colleagues (2011) define teacher noticing as the attentional work of teachers: what they attend to and how they reason about what they observe.

backgrounds to collaborate with each other or work with local communities to understand the foundational culture of their students and key levers of engagement.

- Research-based educational interventions such as curricula and standards based on white middle-class norms can actively inhibit learner achievement and suppress cultural vitality and the vibrancy of diverse languages and community traditions (Nelson Barber & Johnson, 2019). Understanding local contexts and incorporating the opinions of interest-holders may provide a stronger measurement of students' learning progress and be more responsive to the particular needs and dispositions of Indigenous learners.
- Centering Indigenous approaches to education by having school leaders and teachers who are themselves Indigenous may help schools embrace local culture, hold students to elevated expectations, make connections to their local lives, and support mathematics and reading achievement and high school graduation (McKinley et al. (2018).
- Pedagogy emphasizing "student collaboration in small groups to jointly create meaningful products while engaging in content-related dialogue" was found to support students' knowledge retention and attitudes toward mathematics (Hilberg et al., 2000).
- The Cognitively Guide Instruction approach (Carpenter et al., 2014)—including pair or group problem-solving and solution sharing, generous instruction time, and using student thinking and accomplishments—was identified by North Dakota educators as having strong potential for their American Indian students (Apthorp, 2004). Teachers described their American Indian students as reporting increased confidence as problem solvers, increased excitement about learning mathematics, improved verbal skills for presenting in front of groups, and better understanding of numbers.

Attend to students' social–emotional well-being.

- A further strand of related research might address the socio–emotional well-being of students as it relates to mathematics teaching and learning. Especially during mathematics focused on social justice issues, it may be important for educators to attend to students' emotional responses (Kokka, 2022) or social and emotional well-being (Nelson-Barber & Trumbull, 2015).

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Appendix 1: Methods

This scan was conducted for technical assistance purposes and, as such, was not intended to meet What Works Clearinghouse standards of review, but rather to identify recent peer-reviewed research that supports the understanding of issues related to the development and implementation of process standards. Below we list information on how resources were identified. The annotated bibliography includes the identification method for each resource included.

Keywords and search strings used in the search

Theme 1: "standards for mathematics practice," "process standards," and "mathematical proficiency;" "connections," together with each of these terms, also identified sources

Theme 2: "conceptual knowledge," "procedural knowledge," "conceptual understanding," and "procedural understanding" each searched separately

Theme 3: "impacts," "effectiveness," or "practices" plus the keywords included in Theme 2

Theme 4: "equity," "Native American," "American Indian," or "Indigenous" plus the keywords listed in Theme 2

Search of databases

Google Scholar, ERIC

Criteria for inclusion

When reviewing resources, we considered four main factors:

Date of the publication

The most current information includes data within the date range of 2010–present and captures publications within the last 13 years. However, there are several key publications cited prior to 2010 that are included to provide historical context.

Student age/grade range

The most current information includes data for students in Grades K–12.

Source and funder of the report/study/brief/article

Priority is given to IES, nationally funded, and certain other vetted sources known for strict attention to research protocols. However, in an effort to identify relevant studies, the search was not limited to peer-reviewed journals, but instead also included gray literature including conference presentation, technical reports, and online research materials that are based on empirical and/or peer-reviewed research and professional development or practical resources based on empirical research (Koestler et al., 2013).

Methodology

Sources include randomized controlled trial studies, surveys, self-assessments, literature reviews, professional development resources, and policy briefs. Priority for inclusion generally applies to randomized controlled trial study findings, with readers considering the number of participants, sample selections, and sample representations as factors when basing decisions on these resources.

Appendix 2: Annotated Bibliography of Empirical Research Contributing to the Body of Evidence, 2010–Present, Keyed to Primary Review Theme

3–Barajas-López, F., & Larnell, G. V. (2019). Unpacking the links between equitable teaching practices and standards for mathematical practice: Equity for whom and under what conditions? *Journal for Research in Mathematics Education*, *50*(4), 349–361.

This research commentary discusses the interface between equity and standards-setting in mathematics education. The authors provide a framework that relates the CCSSM to equitable teaching practices, and discuss tensions related to co-opting the standards for equity purposes. They also discuss the role of racialized rhetoric and nondominant family and community knowledge.

3–Bartell, T., Wager, A., Edwards, A., Battey, D., Foote, M., & Spencer, J. (2017). Toward a framework for research linking equitable teaching with the standards for mathematical practice. *Journal for Research in Mathematics Education*, *48*(1), 7–21.

The Common Core State Standards for Mathematics (CCSSM) make the standards a goal for all students and, thus, a mechanism for achieving equitable math instruction. Still, in this commentary, the authors suggest that past reform efforts demonstrate that standards without explicit (or companion) teaching practices, and teaching practices without explicit attention to equity, will inevitably result in the failure of the standards to achieve goals for students. The article provides a framework for future research that hypothesizes research-based equitable mathematics teaching practices in support of the CCSSM's Standards for Mathematical Practice, connecting research, policy, and practice to realize the equity potential of the CCSSM.

2–Barbieri, C., & Booth, J. L. (2016). Support for struggling students in algebra: Contributions of incorrect worked examples. *Learning and Individual Differences*, *48*, 36–44.

Middle school algebra students ($N = 125$) randomly assigned within a classroom to a problem-solving control group, a correct worked examples control group, or an incorrect worked examples group completed an experimental classroom study to assess the differential effects of incorrect examples versus the two control groups on students' algebra learning, competence expectancy, and sense of belonging to math class. The study also explored whether prior knowledge impacted the effectiveness of the intervention. A greater sense of belonging and competence expectancy predicted greater learning overall. Students' sense of belonging to math and competence expectancies were high at the start of the study and did not increase as a result of the intervention. A significant interaction between prior knowledge and incorrect worked examples on post-test scores revealed that students with low prior knowledge who

struggle with learning math benefit most from reflecting on highlighted errors within an incorrect worked examples intervention.

2–Belenky, D. M., & Schalk, L. (2014). The effects of idealized and grounded materials on learning, transfer, and interest: An organizing framework for categorizing external knowledge representations. *Educational Psychology Review*, *26*(1), 27–50.

Research in both cognitive and educational psychology has explored the effect of diverse types of external knowledge representations (for example, manipulatives, graphical and pictorial representations, and texts) on a variety of important outcome measures. This paper provides a review of the existing literature on learning with abstract and concrete representations.

2–Bisra, K., Liu, Q., Nesbit, J. C., Salimi, F., & Winne, P. H. (2018). Inducing self-explanation: A meta-analysis. *Educational Psychology Review*, *30*, 703–725.

Self-explanation is a process by which learners generate inferences about causal connections or conceptual relationships. The article describes meta-analysis conducted on research that investigated learning outcomes for participants who received self-explanation prompts while studying or solving problems. The authors conducted a systematic search of relevant bibliographic databases, identifying 69 effect sizes (from 64 research reports) which met inclusion criteria. The overall weighted mean effect size using a random effects model was $g = 0.55$. Authors coded and analyzed 20 moderator variables including type of learning task (for example, solving problems, studying worked problems, and studying text), subject area, level of education, type of inducement, and treatment duration. The authors found that self-explanation prompts are a potentially powerful intervention across a range of instructional conditions. Due to the limitations of relying on instructor-scripted prompts, authors recommend that future research explore computer generation of self-explanation prompts.

2–Booth, J. L., Begolli, K. N., & McCann, N. (2016, November). *The effect of worked examples on student learning and error anticipation in algebra*. [Paper presentation]. Psychology of Mathematics Education–North America Meeting, Tucson, AZ, United States.

Although there is a growing consensus that students can learn effectively from explaining errors, few textbooks include incorrect examples. Since teachers have limited time and resources to consistently generate examples, some have thought to utilize student-generated examples. This study aimed to replicate previous results showing the effectiveness of using worked examples with self-explanation prompts towards investigating growth in conceptual and procedural knowledge. Secondly, this study aimed to examine students' ability to anticipate the types of errors other students might make when solving algebraic equations. The researchers employed a quasi-experimental design where 75 Algebra I students were assigned to treatment and control groups according to their rostered sections. The treatment class was given

a modified subset of practice problems that each included a correct, incorrect, or partial example of that problem's solution. Students completed pre- and post-test measures of conceptual understanding, procedural problem-solving skill, and error anticipation. Results indicate the example-based textbook assignments increased students' procedural knowledge and their ability to anticipate errors one might make when solving problems.

3–Carpenter, T. P., Fennema, E., Franke, M. L., Levi, L., & Empson, S. B. (2014). *Children's Mathematics: Cognitively Guided Instruction*. Heinemann.

The first edition of Children's Mathematics helped teachers understand children's intuitive mathematical thinking and use that knowledge to help children learn mathematics with understanding. This edition provides new insights about Cognitively Guided Instruction based on the authors' research and experience in CGI classrooms over the last 15 years. Highlights include

- how children solve problems using their intuitive understanding of addition, subtraction, multiplication, and division;
- the development of children's mathematical thinking throughout the primary grades;
- instructional practices that promote children's active engagement in mathematics; and
- connections between children's strategies and powerful mathematical concepts.

A new expanded collection of over 90 online video episodes illustrates children's mathematical thinking, interactions between students and teachers, and classroom instruction that builds on children's mathematical thinking. The second edition and videos provide a detailed research-based account of the development of children's mathematical thinking and problem-solving, and how teachers can promote this development in ways that honor children's thinking.

2–Common Core Standards Writing Team. (2023). *Progressions for the Common Core State Standards for Mathematics*. Institute for Mathematics and Education, University of Arizona. https://mathematicalmusings.org/author/wgmccallum/

The 2010 Common Core State Standards in mathematics began with narrative documents describing the progression of a topic across a number of grade levels, informed both by educational research and the structure of mathematics. Those documents were then organized into grade level standards, and subsequent work focused on refining and revising the grade level standards rather than refining the progressions documents. The 2023 Progressions for the Common Core State Standards are updated versions of the earlier progressions drafts, revised and edited to correspond with the standards by members of the original progressions work team, together with other mathematicians, statisticians, and education researchers not involved in the initial writing. They note key connections among standards, point out cognitive difficulties and pedagogical solutions, and provide additional detail. The progressions also provide additional resources for curriculum that illustrates the range and types of mathematical work described by the standards; discussions of individual standards; classroom tasks; teacher professional development; and understanding the importance modeling and language in students' mathematical development.

2–Davenport, J. L., Kao, Y. S., Matlen, B. J., & Schneider, S. A. (2020). Cognition research in practice: Engineering and evaluating a middle school math curriculum. *The Journal of Experimental Education*, *88*(4), 516–535.

This study sought to investigate the effectiveness of the following recommendations proposed in cognitive and learning sciences by (1) facilitating mapping between visual representations, (2) prompting for explanation of worked examples, (3) using quizzing to promote learning, and (4) spacing practice opportunities over time. The authors describe a large-scale, cluster-randomized trial study that used these four principals to redesign a middle school math curriculum to test the efficacy of these practices. This study implemented multidimensional item response theory to estimate student math achievement on outcome measures. The results of the study showed trends of an overall positive impact for the redesigned curriculum, however, statistical significance was not found and may have been tempered likely due to high attrition numbers.

3–Gutiérrez, R. (2022). A spiritual turn: Toward desire-based research and Indigenous futurity in mathematics education. *Journal for Research in Mathematics Education*, *53*(5), 379–388. https://doi.org/10.5951/jresematheduc-2022-0005

This research commentary makes an argument for desire-based research frameworks and Indigenous futurity praxis as key components of a spiritual turn in mathematics education research, distinct from the prior sociopolitical turn in the field. By futurity, the author means, in part, the practice within Indigenous communities of the past, present, and future being intertwined. As futurity becomes more popular in mainstream venues, it raises questions about how it will affect mathematics education (research). The author analyzes equity issues arising in the March 2022 issue of JRME, raising questions to consider in research and offers suggestions making it possible to embrace a spiritual turn.

2–Hurrell, D. (2021). Conceptual knowledge OR procedural knowledge OR conceptual knowledge AND procedural knowledge: Why the conjunction is important to teachers. *Australian Journal of Teacher Education*, *46*(2), 57–71.

This literature review was conducted in response to notions suggested by some teachers that conceptual and procedural knowledge were mutually exclusive and that conceptual knowledge should be prioritized over procedural knowledge. This paper examines literature regarding conceptual and procedural knowledge, and their place in the classroom, to offer teachers and teacher educators' advice on some of the more pressing issues and understandings around them. The paper includes a synthesis of extant and seminal literature to advise teachers and teacher educators on how a deeper insight into conceptual and procedural knowledge could improve the quality of mathematics teaching. In particular, the review summarizes the complexities in

defining conceptual and procedural knowledge, a consensus in the importance of both, and the challenge of separating or determining the ordering of conceptual and procedural knowledge.

2–Kapur, M. (2014). Productive failure in learning math. *Cognitive Science*, 38(5), 1008–1022. https://onlinelibrary.wiley.com/doi/pdfdirect/10.1111/cogs.12107

This study investigated the question: When learning a new math concept, should learners be first taught the concept and its associated procedures and then solve problems, or solve problems first even if it leads to failure and then be taught the concept and the procedures? Two randomized-controlled studies found that both methods lead to elevated levels of procedural knowledge. However, students who engaged in problem-solving before being taught demonstrated significantly greater conceptual understanding and ability to transfer to novel problems than those who were taught first. The second study further showed that when given an opportunity to gain experience from the failed problem-solving attempts of their peers, students outperformed those who were taught first, but not those who engaged in problem-solving first. Process findings showed that the number of student-generated solutions significantly predicted learning outcomes. These results challenge the conventional practice of direct instruction to teach new math concepts and procedures and propose the possibility of learning from one's own failed problem-solving attempts or those of others before receiving instruction as alternatives for better math learning.

2–Kelemanik, G., Lucenta, A., & Creighton, S. J. (2016). *Routines for reasoning: Fostering the mathematical practices in all students*. Heinemann.

The authors describe four classroom routines designed to help students develop their mathematical thinking skills. The four routines, aligned with the CCSS Standards for Mathematical Practice, are (1) Capturing Quantities: encouraging abstract and quantitative reasoning; (2) Connecting Representations: noticing and using mathematical structure; (3) Recognizing Repetition: developing repeated reasoning skills; and (4) Three Reads: starting and sustaining thinking in problem-solving situations. Each routine is organized around a structure that supports repeated use of the thinking skills to employ and the questions to ask until the steps to follow become automatic, enabling all students to engage more fully in learning opportunities while building crucial mathematical thinking habits.

3–Kokka, K. (2022). Toward a theory of affective pedagogical goals for social justice mathematics. *Journal for Research in Mathematics Education*, *53*(2), 133–153. https://doi.org/10.5951/jresematheduc-2020-0270

This article aims to articulate how three interrelated pedagogical goals—affective, critical, and dominant—should be considered, when teaching social justice mathematics (SJM), to take into account what students would do or how they would feel during SJM lessons. Two illustrative cases, one in a Title I public middle school and the other in an elite independent school, are presented to explore how affective pedagogical goals may be mediated by context.

1–Koestler, C., Felton-Koestler, M., Bieda, K., & Otten, S. (2013). *Connecting the NCTM process standards and the CCSSM practices*. National Council of Teachers of Mathematics.

The focus of this book is on the Standards of Mathematical Practice (SMPs) outlined in the Common Core State Standards in Mathematics. The book describes how the SMPs are not new but linked to previous practices and standards articulated by other groups, including the National Council of Teachers of Mathematics (NCTM). For example, problem-solving and reasoning are at the core of all practices outlined in CCSSM, just as they have been at the core of NCTM's vision for mathematics education since the publication of *An Agenda for Action (1980s)* in 1980. Subsequent NCTM curriculum recommendations emphasized and elaborated the role and place of mathematical processes and practices. The book also explores the SMPs in greater detail, reaffirming the importance to learning mathematics of habits of mind, mathematical processes, and proficiency.

2–Lambert, R., & Sugita, T. (2016). Increasing engagement of students with learning disabilities in mathematical problem-solving and discussion. *Support for Learning*, *31*, 347–366.

This research review describes a gap in the literature surrounding engagement of students with learning disabilities (LD) in standards-based mathematical classrooms. The review found that students with LD were supported towards equal engagement in standards-based mathematics through multi-modal curriculum, consistent routines for problem-solving, and teachers trained in mathematical knowledge for teaching. Using this small set of studies (7), the authors identify the need to deepen the engagement of students with LD in mathematical problem-solving and discussion. This review concludes with implications for teaching and learning.

2–Lehtinen, E., Hannula-Sormunen, M., McMullen, J., & Gruber, H. (2017). Cultivating mathematical skills: from drill-and-practice to deliberate practice. *ZDM—Mathematics Education*, *49*, 625–636. https://doi.org/10.1007/s11858-017-0856-6

Research has found that the relationship between conceptual and procedural knowledge in mathematics is bidirectional (Rittle-Johnson et al., 2015) and that there may be inter-individual differences in the reliance of procedural or conceptual knowledge in developing high-level mathematical knowledge (Hallett et al., 2010). The authors posit that if we view procedural fluency in mathematics as complex cognitive processes, then practices that purport to support procedural fluency should align with attempts to develop conceptual knowledge. Drawing on expertise research, the authors explore the concept of deliberate practice in the development of high-level expertise; deliberate practice is widely accepted to be a key factor explaining exceptional performance in sport, music, and different professional fields (Ericsson et al., 2006). In this article, the authors review the aspects of deliberate practice that could be relevant for rethinking the role and nature of practice in mathematics education.

3–Louie, N., & Zhan, W.- Y. (2022). A socio-ecological framework for research in mathematics education. *Journal for Research in Mathematics Education*, *53*(5), 365–371.

The authors provide a research commentary in response to four articles published in the March 2022 issue of the *Journal for Research in Mathematics Education*, including van Es et al. (2022), Kokka (2022), and Brantlinger (2022). The authors aim to make connections across mathematics education research, including studies that center equity and those that do not, using a socio–ecological framework that emphasizes the multiple layers of activity in the field to better understand equity in mathematics education. Although researchers may acknowledge the multiple layers of social activity impacting equity efforts, some may choose to focus on a more individual, microlevel (van Es et al., 2022; Kokka, 2022) while others focus on a broader, macrolevel (Brantlinger, 2022). The authors of this article describe multiple implications stemming from their framework, including the need for research to account for the fact that power flows through all social activity, even in research that does not directly center equity. Studies that may focus on more microlevel interactions may need to further situate findings and recommendations within larger macrolevel historical contexts and vice versa.

3–Matthews, P. G., Herbst, P., Crespo, S., & Lichtenstein, E. K. (2022). Continuing a conversation about equity-focused research in mathematics education. *Journal for Research in Mathematics Education*, 53(5), 342–349.

In this editorial, the authors provide an overview of the *Journal for Research in Mathematics Education* (JRME) issue that includes research and commentaries responding to a prior issue on equity in mathematics education. This editorial synthesizes the arguments made in the three commentaries of the November 2022 JRME issue as part of an effort to explore how the conversation about equity in mathematics education could continue beyond the publishing of these commentaries. The editorial authors pose three problems for equity-focused research in mathematics education: (1) the extent to which an equity focus belongs on a distinct branch of mathematics education research, (2) the tensions between theorists philosophizing what is needed for equity and advocates pursuing action-oriented processes towards attaining equity, and (3) situating equity in place and time with the work of an international research community.

2–McCoy, G. D. (2023). *The effect of the three reads strategy on the story problem solving skills of middle school students who are deaf/hard of hearing*. Minot State University ProQuest Dissertations Publishing. https://www.proquest.com/openview/8f586723543858c44fd2bbdeea37f57e/1.pdf?pq-origsite=gscholar&cbl=18750&diss=y

This study examined the effectiveness of the Three Reads strategy for deaf and hard of hearing (DHH) middle school students using it to solve arithmetic story problems. On standardized and state assessments, a majority of students who are DHH have low scores on arithmetic story problems, and many teachers are looking for effective

strategies to improve their students' math story problem-solving skills. The literature offered limited strategies to teach math story problems to students who are DHH; however, the Three Reads strategy is effective for other, distinct types of students, and the possibility of its effectiveness with students who are DHH is plausible. Six middle school deaf students attending the state School for the Deaf in a midwestern state were selected to participate in this pre-and post-test study to examine the impact of the Three Read strategy on the students' math story problem-solving skills. The results showed that all students improved their scores after being taught to use the Three Reads strategy; students reading on grade level benefited the most. Directions and recommendations for further research tied to teaching story problems are discussed.

3–McKinley, B., Brayboy, J., & Lomawaima, K. T. (2018). Why don't more Indians do better in school? The battle between U.S. schooling & American Indian/Alaska Native education. *Daedalus*, *147*(2), 82–94. https://doi.org/10.1162/DAED_a_00492

The authors studied an Alabama elementary school in which Indigenous education and Indigenous schooling are woven together to better serve Native peoples. The school's leaders and teachers are products of the school and community, a factor which helps the school embrace local culture, holds students to elevated expectations, and makes connections to their local lives. At the school, 100 percent of students met the math standards for Alabama, and 91 percent met reading standards (79 percent of those at an advanced level) as compared to their Indigenous peers across the state (where academic achievement for American Indian children is among the lowest of all students). Authors reported that 91 percent of the students graduate from high school.

3–Melhuish, K., Thanheiser, E., White, A., Rosencrans, B., Shaughnessy, J. M., Foreman, L., Riffel, A., & Guyot, L. (2022). The efficacy of research-based "mathematics for all" professional development. *Journal for Research in Mathematics Education*, *53*(4), 307–333.

This article discusses what makes mathematics professional development (PD) successful and in what ways. The authors share a research-based PD model that was implemented in elementary schools in an urban school district for 3 years. The model uses a pseudo-lesson study approach and emphasizes standards-based instruction. The research found that teachers made gains in knowledge and instruction quality. However, whereas some students saw gains on standardized assessments, this was the case only for students who were not members of historically minoritized groups (Black/Latinx), countering assumptions that the PD would lead to equitable achievement results. The authors conclude with a discussion of how a colorblind approach to PD may account for the inequitable results.

2–Mertens, U., Finn, B., & Lindner, M. A. (2022). Effects of computer-based feedback on lower- and higher-order learning outcomes: A network meta-analysis. *Journal of Educational Psychology, 114*(8), 1743–1772. https://doi.org/10.1037/edu0000764

Feedback is one of the most crucial factors for successful learning. Computer-based learning and testing environments enable automated feedback. Previous meta-analyses suggest that diverse types of feedback are not equally effective. Following an extensive literature search, the paper compared classical feedback variations such as *Knowledge of Results* (KR), *Knowledge of Correct Response* (KCR), *Elaborated Feedback* (EF), and *Answer-Until-Correct* (AUC) feedback, with each other and with a *No Feedback* (NoFB) control group. Our findings indicate that elaborated feedback (EF) is likely to be the most effective for both lower-order (for example, recall and recognition) and higher-order (for example, transfer) learning outcomes compared with the other feedback variants. Studies of feedback that shared the correct response (KCR) or allowed students to answer until they got the correct answer (AUC) found small to large effect sizes on learning outcomes. Knowledge of the results (that is, whether they were correct or not) was less effective than the other feedback types on improving lower-order and higher-order learning outcomes. Several subgroup analyses are reported to identify moderating factors for the effectiveness of different feedback interventions for different learner characteristics (for example, sample source and prior knowledge level) and test characteristics (for example, learning domain and test format).

1–National Assessment Governing Board. (2020). *Governing board approves updated mathematics framework for the 2025 National Assessment of Educational Progress*. [Press release]. https://www.nagb.gov/news-and-events/news-releases/2019/release-20191121-governing-board-approves-updates-mathematics-framework.html

This press release describes the introduction of five mathematical practices as the most meaningful change for the 2025 NAEP Mathematics Assessment. Reflecting decades of efforts to more clearly specify mathematical processes like "higher-order thinking" and "mathematical reasoning," the practices provide a richer picture of student achievement and engagement processes needed to do the work of mathematics. The mathematics practices are closely intertwined with the content objectives.

1–National Assessment Governing Board. (2022). *Mathematics assessment framework for the 2022 and 2024 National Assessment of Educational Progres*s. https://www.nagb.gov/content/dam/nagb/en/documents/publications/frameworks/mathematics/2022-24-nagb-math-framework-508.pdf https://www.nagb.gov/

The report provides a history of the National Assessment for Educational Progress and describes the framework (i.e., the design of the assessment) for the 2022 and 2024 years. The discussion of mathematical content and complexity provides additional detail on item formats and assessment design.

2–National Council of Teachers of Mathematics. (2023). *Reasoning and decision-making, not rote application of procedures position.* https://www.nctm.org/Standards-and-Positions/Position-Statements/Procedural-Fluency-in-Mathematics/

This position statement presents four declarations related to ensuring that every student has access to and develops procedural fluency, which the authors argue is an essential component of equitable teaching and is necessary to developing mathematical proficiency and mathematical agency. The declarations are that each and every student must have access to teaching that (1) connects concepts to procedures, (2) explicitly develops a reasonable repertoire of strategies and algorithms, (3) provides substantial opportunities for students to learn to choose from among the strategies and algorithms in their repertoire, and (4) implements assessment practices that attend to all components of fluency. These declarations apply to computational fluency across the K–12 curriculum, including basic facts, multidigit whole numbers, and rational numbers, as well as to other procedures throughout the curriculum such as comparing fractions, solving proportions or equations, and analyzing geometric transformations.

1–National Research Council. (2012). *Education for life and work: Developing transferable knowledge and skills in the 21st century*. The National Academies Press.

This report describes a set of key skills that increase deeper learning, college and career readiness, student-centered learning, and higher-order thinking. These include both cognitive and non-cognitive skills such as critical thinking, problem-solving, collaboration, effective communication, motivation, persistence, and learning to learn. Twenty first century skills also include creativity, innovation, and ethics that are important to later success and may be developed in formal or informal learning environments. Skills and knowledge such as the following are discussed: constructing and evaluating evidence-based arguments; non-routine problem-solving; complex communication; disciplinary discourse; systems thinking; critical thinking; motivation, persistence; collaboration and teamwork; and self-regulation. This report also describes how these skills relate to each other and to more traditional academic skills and content in the key disciplines of reading, mathematics, and science. The report summarizes the findings of research that investigates the importance of such skills to success in education, work, and other areas of adult responsibility and that demonstrates the importance of developing these skills in K–16 education. The report also discusses features related to learning these skills, such as teacher professional development, curriculum, assessment, after-school and out-of-school programs, and informal learning centers such as exhibits and museums.

3–Nelson-Barber, S., & Trumbull, E. (2015). *The Common Core initiative, education outcomes, and American Indian/Alaska Native Students: Observations and recommendations.* The Center on Standards and Assessments Implementation. WestEd.

This monograph explores the ways in which large-scale school reform efforts play out in American Indian/Alaska Native (AI/AN) communities and schools that serve significant numbers of AI/AN students. The authors situated their investigation of the topic within a historical and cultural perspective, drawing on an extensive body of research. Drawing on that research, they identified specific, productive steps that can

be undertaken to select and implement education reforms in ways that ensure AI/AN student academic success and personal well-being.

3–Nelson-Barber, S., & Johnson, Z. (2019). Raising the standard for testing research-based interventions in Indigenous learning communities. *International Review of Education*, *65*(1), 47–65.

Drawing on a qualitative research study, conducted over three years with practicing Diné educators from across the Navajo Nation, Nelson-Barber & Johnson (2019) described how curricula and standards based on White middle-class norms may flatten the vibrancy of diverse languages and community traditions. The authors demonstrated that, for distinctive populations like Indigenous groups, research-based educational interventions can actively inhibit learner achievement and suppress cultural vitality; so-called "best practices" may not be best for all. The authors provided examples of ways in which "understanding context" is an essential ingredient paving the way for student success. They also asserted that in order to optimize the potency of educational innovations developed for Indigenous learners, interventions must adhere to a higher standard of assessment practice. Local testing and incorporating the opinions of interest-holders in the community may provide a stronger measurement of students' learning progress and, thus, be more responsive to the particular needs and dispositions of Indigenous learners.

2–Rittle-Johnson, B., Fyfe, E. R., & Loehr, A. M. (2016). Improving conceptual and procedural knowledge: The impact of instructional content within a mathematics lesson. *British Journal of Educational Psychology*, *86*(4), 576–591.

This randomized experiment with 180 Grade 2 children evaluated the effect of instruction on a math concept and procedure within the same lesson relative to a comparable amount of instruction on the concept alone. Children received a classroom lesson on mathematical equivalence in one of four conditions that varied in instruction type (conceptual or combined conceptual-and-procedural) and in instruction order (instruction before or after solving problems). Children who received two iterations of conceptual instruction had better retention of conceptual and procedural knowledge than children who received both conceptual and procedural instruction in the same lesson. Order of instruction did not impact outcomes. The results suggest that within a single lesson, spending more time on conceptual instruction may be more beneficial than time spent teaching a procedure when the goal is to promote more robust understanding of target concepts and procedures.

2–Rittle-Johnson, B., & Schneider, M. (2015). Developing conceptual and procedural knowledge of mathematics. In R. C. Kadosh & A. Dowker (Eds.), *Oxford handbook of numerical cognition* (pp. 1102–1118). Oxford University Press.

This chapter reviews recent studies on the relationship between conceptual and procedural knowledge in mathematics and highlights examples of instructional methods for supporting both types of knowledge. It concludes with prominent issues to address in future research, including gathering evidence for the validity of measures of conceptual and procedural knowledge and specifying more comprehensive models for how conceptual and procedural knowledge develop over time.

2–Rittle-Johnson, B., Schneider, M., & Star, J. R. (2015). Not a one-way street: Bidirectional relations between procedural and conceptual knowledge of mathematics. *Educational Psychology Review, 27*, 587–597.

The authors' review of the empirical evidence for mathematics learning indicates that procedural knowledge supports conceptual knowledge, as well as vice versa, and, thus, the relationship between the two types of knowledge is bidirectional. This consensus about the bidirectionality of these types of knowledge holds, despite the fact that alternative orderings of instruction on concepts and procedures have rarely been compared, and there is limited empirical support for one ordering of instruction over another. The authors consider possible reasons why mathematics education researchers often believe that a conceptual-to-procedural ordering of instruction is optimal and why so little research has evaluated this claim. The authors underscore the need for empirical research on the effectiveness of varying ways to sequence instruction on concepts and procedures.

2–Siegler, R., Carpenter, T., Fennell, F., Geary, D., Lewis, J., Okamoto, Y., Thompson, L., & Wray, J. (2010). *Developing effective fractions instruction for kindergarten through 8th grade: A practice guide* (NCEE 2010-4039). National Center for Education Evaluation and Regional Assistance, Institute of Education Sciences, U.S. Department of Education. whatworks.ed.gov/ publications/practiceguides

This practice guide provides research-based recommendations to improve students' understanding of fraction concepts in kindergarten through Grade 8. The practice guide includes strength ratings for each recommendation, which describe the extent to which there is strong, generalizable evidence supporting the recommendations. This guide presented five recommendations to develop young children's understanding of early fraction concepts and ideas for helping older children understand the meaning of fractions and the computations involved. The guide describes ways to build on students' existing strategies to solve problems involving ratios, rates, and proportions. The five recommendations are (1) build on students' informal understanding of sharing and proportionality to develop fraction concepts; (2) help students recognize that fractions are numbers and that they expand the number system beyond whole numbers; (3) help students understand why procedures for computations with fractions make sense; (4) develop students' conceptual understanding of strategies for solving ratio, rate, and proportion problems before exposing them to cross-multiplication as a procedure to solve such problems; and (5) professional development programs should place a high priority on improving teachers' understanding of fractions and how to teach them. Of these recommendations, recommendations 1, 4, and 5 received a lower-tiered IES rating of 4 with minimal

evidence supporting these recommendations. Recommendations 2 and 3 received an IES rating of 3 with a moderate level of evidence supporting these recommendations.

2–Sinha, T., & Kapur, M. (2021). When problem solving followed by instruction works: Evidence for productive failure. *Review of Educational Research, 91*(5), 761–798.

This paper examines the question: When learning a new concept, should students engage in problem-solving followed by instruction (PS-I) or instruction followed by problem-solving (I-PS)? The authors report evidence from a meta-analysis of 53 studies with 166 comparisons that compared PS-I with I-PS design. Their results showed a significant, moderate effect in favor of PS-I (Hedge's g =0.36 [95% confidence interval 0.20; 0.51]). The effects were even stronger (Hedge's g ranging between 0.37 and 0.58) when PS-I was implemented with high fidelity to the principles of productive failure (PF), a subset variant of PS-I design. Students' grade level, intervention time span, and its (quasi-) experimental nature contributed to the efficacy of PS-I over I-PS designs. Contrasting trends were observed for younger age learners (Grade 2-5) and for the learning of domain-general skills, for which effect sizes favored I-PS. Overall, an estimation of true effect sizes after accounting for publication bias suggested a strong effect size favoring PS-I (Hedge's g =0.87).

2–Star, J. R., Caronongan, P., Foegen, A., Furgeson, J., Keating, B., Larson, M. R., Lyskawa, J., McCallum, W. G., Porath, J., & Zbiek, R. M. (2015). *Teaching strategies for improving algebra knowledge in middle and high school students.* (NCEE 2015-4010). National Center for Education Evaluation and Regional Assistance, Institute of Education Sciences, U.S. Department of Education.

This practice guide provides research-based recommendations to improve algebra knowledge. These include (1) use solved problems to engage students in analyzing algebraic reasoning and strategies; (2) teach students to utilize the structure of algebraic representations; (3) teach students to intentionally choose from alternative algebraic strategies when solving problems. The first two recommendations were only found to have minimal evidence, while the third had moderate evidence from the literature. Research studies reviewed and used to develop these recommendations showed positive effects on students' conceptual knowledge and problem-solving capabilities. Studies supporting the third recommendation found that teaching alternative strategies can improve procedural flexibility that comes with developing procedural knowledge. In alignment with the National Mathematics Advisory Panel definition of proficiency, the authors remind readers that "Conceptual knowledge, procedural knowledge, and procedural flexibility are distinct competencies. Mathematical proficiency results when children develop these and other competencies and form links between them." They also determined that existing measures may not demonstrate a distinction between these diverse types of knowledge. A definition for these three domains is included in the practice guide.

3–van Es, E. A., Hand, V., Agarwal, P., & Sandoval, C. (2022). Multidimensional noticing for equity: Theorizing mathematics teachers' systems of noticing to disrupt inequities.

Journal for Research in Mathematics Education, *53*(2), 114–132. https://doi.org/10.5951/jresematheduc-2019-0018

Teacher noticing of classroom, referring to teachers' attention to and sensemaking of classroom activity, shapes who is invited to participate and whose forms of knowing are included in mathematics classrooms. Prior studies on teachers' noticing have described the challenges in understanding the skills needed by teachers to enact practices aimed at creating more affirming classroom environments for minoritized students and how to make sense of instructible moments. The authors introduce a multidimensional framework for noticing equity centered around the nations of stretch and expanse. The authors illustrate the multidimensionality of teachers' noticing by drawing on data from two mathematics teachers, using the data to illuminate how instructional decisions and noticing are informed by teachers' awareness and understanding of their students' cultures, histories, and communities.

2–Woodward, J., Beckmann, S., Driscoll, M., Franke, M., Herzig, P., Jitendra, A., Koedinger, K. R., & Ogbuehi, P. (2012). *Improving mathematical problem solving in Grades 4 through 8*: *A practice guide* (NCEE 2012-4055). National Center for Education Evaluation and Regional Assistance, Institute of Education Sciences, U.S. Department of Education.

This IES practice guide, geared toward educators who want to improve students' mathematical problem-solving, provides five recommendations for improving students' mathematical problem-solving in Grades 4 through 8. These recommendations include (1) prepare both routine and non-routine problems and use them in whole-class instruction; (2) assist students in monitoring and reflecting on the problem-solving process; (3) teach students how to use visual representations; (4) expose students to multiple problem-solving strategies; and (5) help students recognize and articulate mathematical concepts and notation. The authors define and discuss three outcome domains related to these recommendations, including procedural knowledge, conceptual understanding, and procedural fluency.

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